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MAGNETO-FLUID DYNAMICS OF
BODIES IN ALIGNED FIELDS

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ABSTRACT

The motion of a perfectly conducting fluid past a fixed body in the presence of an aligned magnetic field can be regarded as the limit of a number of problems viz: (i) when the fluid is a finite conductor, (ii) an unsteady problem, (iii) the magnetic and velocity fields are not parallel at infinity. The limits of these more general problems agree in predicting a force on the body and on the existence of upstream and downstream wakes. However, if the magnetic field is fairly strong (i) does not predict a downstream wake while (ii) and (iii) do.

An attempt to reconcile these limit results is made here. In the first part we show that the unsteady problem associated with a finitely conducting fluid and an aligned field is non-unique but that by making an appeal to the theory of real fluids the conclusion of (i) and (ii) can be obtained as special cases of the general solution. In the second part the steady problem, assuming that the conductivity is finite and the magnetic field is nearly aligned, is considered and it is shown that the flow fields predicted in (i) and (iii) are limiting cases of this more general problem. The manner of the changeover from the solution for (i) to that for (iii) is elucidated.

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1. Introduction

The properties of the two-dimensional motion of an incompressible

inviscid fluid, of electrical conductivity σ , past a fixed non-conducting

body and in the presence of an imposed magnetic field have attracted much

interest in recent years. In particular, the properties of the motion when

the magnetic and velocity fields are aligned, i. e. parallel, at large distances

from the body and σ is infinite have been the subject of some controversy.

Sears and Resler (1) pointed out that in these circumstances there is a

simple solution of the equations of motion in which the magnetic field

outside the body is everywhere parallel to the velocity field which in turn

is identical with that when $\sigma = 0$. Inside the body the magnetic field

vanishes. One important deduction from this solution is that the drag on

the body is zero. They also briefly discussed the solution when σ is

large but not finite and concluded that if the body is thin, their solution

is consistent with it in a limited sense. Subsequently, Ring (2) extended

their solution to include the effects of oscillating a thin body about an axis.

Some doubts about the relevance of these solutions to physically realisable situations were cast by Stewartson (3), who meant that it was doubtful whether the solution obtained by Sears and Resler (1) was the limit of a general fluid dynamic problem in which the motion was started started from rest, at which time the magnetic field was everywhere uniform and at an inclination α to the mean direction of flow, and the fluid was of finite conductivity, as $t \rightarrow \infty$, $\alpha \rightarrow 0$, $\sigma \rightarrow \infty$, either jointly or separately. In support of these doubts he considered the unsteady motion of a thin body with $\sigma = \infty$, $\alpha = 0$, showing that the disturbance produced manifested itself partly as a potential flow and partly in the form of wakes travelling from the body along the lines of magnetic force without being either dispersed or diffused. Let $\beta = H_\infty^2/4\pi\rho V_\infty^2$, where H_∞, V_∞ are the magnitudes of the undisturbed magnetic and velocity fields and ρ is the fluid density: then if $\beta > 1$ these wakes travel upstream and downstream of the body while if $\beta < 1$ both travel downstream. For the simple solution to be correct these wakes must be absent but all that could be deduced with certainty from the temporal solution is that the assumption of small disturbances breaks down at infinite times. It is noted however that the argument in the paper is not entirely satisfactory. At any finite value of t it may be shown that the magnetic field is continuous at the body (taking the permeability of both fluid and body to be unity) because the normal component of the magnetic field is non-zero. In the paper it was tacitly assumed that the same condition

must hold at an infinite time but as M. B. Glauert kindly pointed out to

the author the properties of the steady state solution imply that the normal

component of the magnetic field must be zero if the disturbances are small

so that a discontinuity in the tangential component of the magnetic field is

permissive. If the discontinuity is allowed and the disturbances in the fluid

are kept small the solution obtained by Sears and Resler (1) is recovered,

but of course the disturbance to the magnetic field in the body is not small

since it is actually zero. Consequently, as stated above, the only legitimate

conclusion from a study of the general temporal problem is that the ultimate

steady solution, if it exists, is not uniformly a small perturbation from the

undisturbed state.

There is however a limiting temporal solution which can be carried

out without any such difficulty, namely if $\beta \gg 1$ and here Stewartson (3)

showed that the ultimate solution is quite different from the prediction in

(1) and contains two wakes extending upstream and downstream to infinity.

Thus at least one of these two solutions must be a singular limit. If infinite

wakes are permitted it follows that arbitrary conditions at infinity cannot be

imposed on the flow when $\beta > 1$ without losing physical reality and the

question of determining physically realistic solutions becomes an order of

magnitude harder to answer. We note that the solution with $\beta \gg 1$ is

obtained by neglecting the convection terms in the equations, i.e. by

assuming that the fluid has velocity but does not move!

The second limit problem $\alpha \to 0$ was investigated by Stewartson

(4) assuming a prefectly conducting fluid, and steady conditions. In

fact the general problem when the magnetic field is at an arbitrary angle of

incidence α to the velocity field was solved and on taking the limit $\alpha \to 0$

it was found that the flow properties are in accord with those suggested by

the temporal solution and not with the hypothesis that wakes are absent.

In particular the drag on a symmetric and symmetrically disposed body

takes the form

$$\frac{4\rho V_\infty^2}{\alpha} \int \{f'(x)\}^2 \, dx \qquad\qquad (1.1)$$

where $f'(x)$ is the slope of one side of the body; if the flow properties

were independent of the magnetic field the drag would be zero. However,

this argument is not conclusive since the theory breaks down in the limit

$\alpha \to 0$ because the disturbances are no longer small and because when

$\alpha \ll \epsilon$, where ϵ is a measure of the slope of the body, the wakes

intersect the body.

The third limit problem is to study the steady flow properties

when $\alpha = 0$ but σ is finite with particular reference to the solution

when σ is large. This has been carried out by Lary (5) for a thin

body who found that the drag is proportional to $\sigma^{\frac{1}{2}}$ when σ is large.

Further if $\beta < 1$ a strong wake develops downstream of the body only,

as σ is increased, while if $\beta > 1$ the strong wake develops upstream only.

Since the disturbances produced by the body increases with σ the solution

obtained by Lary (5) is of limited validity, the precise condition being that
$\sigma \, \epsilon^2 \ll 1$, and so cannot be applied immediately to the case $\sigma = \infty$.
However it does present further evidence in support of the view that the
flow of a fluid of infinite conductivity is grossly different from that of a
fluid of zero conductivity, that it contains either downstream or upstream
wakes and that the force on the body is non-zero. Nevertheless the
solutions obtained by the three limiting procedures $t \to \infty$, $\alpha \to 0$, $\sigma \to \infty$
do differ in certain details of which the most important is that in Lary's
solution there is no downstream wake when $\beta > 1$ while in the other
limiting procedures ($t \to \infty$ and $\alpha \to 0$) such a wake is present. In this
paper we shall attempt to reconcile the limiting procedure $\sigma \to \infty$ with the
other two. First we shall discuss the properties of the unsteady linearized
equations in the presence of an aligned magnetic field and show that the
equations do not have a unique solution unless appeal is made to the theory
of real fluids in order to determine the vorticity at the rear of the body. It
is argued that if the flow remains almost uniform and the body is appropriately
streamlined the non-uniqueness may be removed and Lary's solution is then
recovered. In the temporal problem with $\sigma \gg 1$ for which a solution could
be found $\beta \gg 1$ the streamlines are greatly disturbed and there are some
theoretical grounds for accepting the solution in (3) if $\sigma = \infty$ and in (4)
if $\sigma < \infty$.

In the second problem the steady flow properties when $\sigma \gg 1$, $\alpha \ll 1$ but $\sigma \alpha^2$ is finite are studied and the manner of the transformation of the solution from (5) to (6) as $\sigma \alpha^2$ varies from ∞ to 0 is demonstrated. In particular the disappearance of the strong wake downstream of the body as $\sigma \alpha^2 \to 0$ for $\beta > 1$ is elucidated.

Although it seems clear that the simple solution obtained by Sears and Resler (1) is not the limit of any of the more general problems considered here, we are not yet justified in concluding that it is essentially isolated. For there is another limit which has not yet been considered, namely $\nu \to 0$, where ν is the kinematic viscosity of the fluid. In the discussion it has been tacitly assumed that $\nu = 0$ from the outset and it may be that if the order of taking limits is reversed the flows obtained may be quite different. In this connection it is noted that Hasimoto (6) has shown that if $\sigma = \infty$, $\nu \neq 0$, $\beta < 1$, there is a solution of the governing equations which is identical with that for a non-conducting fluid of different viscosity: if $\beta > 1$ the two flows are closely related. Whether Hasimoto's solution is the limit of solutions with finite σ, as $\sigma \to \infty$, is an unresolved question at present but certainly its limit as $\nu \to 0$ is identical with (1) .

PART I

UNSTEADY THEORY

2. The Basic Equations

Consider a cylinder body of characteristic length $\underline{a}$ fixed in

an incompressible fluid of electrical conductivity $\underline{\sigma}$ and permeability

$\underline{\mu = 1}$. The fluid is initially at rest and at time $\underline{t' = 0}$ is set in motion

such that the velocity at an infinite distance upstream of the body changes

abruptly to a magnitude $\underline{V_\infty}$ in a direction perpendicular to the generators

of the body (i.e. to the $\underline{z}$ direction) and subsequently is kept constant.

In addition a magnetic field is imposed which at all finite times is uniform,

of magnitude H_∞ and parallel to the velocity field at infinite distances

upstream of the body. Denoting the velocity and magnetic vectors by

$\underline{q}$, $\underline{H}$ the fluid pressure by $\underline{p}$ and the density by $\underline{\rho}$ the governing

equations are

$$\frac{\partial \underline{H}}{\partial t'} - \operatorname{curl}(\underline{q} \times \underline{H}) + \frac{1}{4\pi\sigma} \operatorname{curl} \operatorname{curl} \underline{H} = 0 , \qquad \operatorname{div} \underline{H} = 0 \qquad (2.1)$$

$$\frac{\partial \underline{q}}{\partial t'} + (\underline{q} \cdot \operatorname{grad})\underline{q} = -\frac{1}{\rho} \operatorname{grad} p + \frac{1}{4\pi\rho}(\operatorname{curl} \underline{H} \times \underline{H}), \qquad \operatorname{div} \underline{q} = 0 . \qquad (2.2)$$

Let $\underline{ax}$, $\underline{ay}$ measure distance along and perpendicular to the direction of the

undisturbed stream and in the plane of motion: let $at'/V_\infty = t$. Further suppose

that the disturbances caused by the body are small so that squares and

products of the perturbations in $\underset{\sim}{q}$, $\underset{\sim}{H}$ may be neglected. Write

$$\underset{\sim}{q} = V_\infty \underset{\sim}{i} + V_\infty \underset{\sim}{v} \, , \qquad \underset{\sim}{H} = H_\infty \underset{\sim}{i} + H_\infty \underset{\sim}{h} \qquad (2.3)$$

where $\underset{\sim}{i}$ is a unit vector in the x direction and on substituting into (2.1),
(2.2) we get

$$\nabla^2 \underset{\sim}{h} - R \left(\frac{\partial \underset{\sim}{h}}{\partial t} + \frac{2 \underset{\sim}{h}}{\partial x} \right) + R \frac{\partial \underset{\sim}{v}}{\partial x} = 0 \, , \quad \frac{\partial \underset{\sim}{v}}{\partial t} + \frac{\partial \underset{\sim}{v}}{\partial x} = \text{grad } P + \beta \frac{\partial \underset{\sim}{h}}{\partial x} \quad (2.4)$$

where

$$R = 4\pi a \sigma V_\infty, \quad \beta = H_\infty^2 \Big/ 4\pi\rho V_\infty^2 \, , \qquad (2.5)$$

$$p = -\rho V_\infty^2 (P + \beta h_x) + p_\infty \qquad (2.6)$$

and p_∞ is the fluid pressure at infinity. The velocity and magnetic fields

may be expressed in terms of two scalar functions ϕ, ψ as follows:

$$v_x = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x \partial t} + \beta \frac{\partial^2 \psi}{\partial x \partial y} \quad , \qquad v_y = \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial t} - \beta \frac{\partial^2 \psi}{\partial x^2} \qquad (2.7)$$

$$h_x = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y \partial t} \quad , \qquad h_y = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial t} \qquad (2.8)$$

$$P = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 \phi - \beta \frac{\partial^2 \phi}{\partial x^2} \qquad (2.9)$$

where

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (2.10)$$

and

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) \nabla^2 \psi - R \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 \psi + R\beta \frac{\partial^2 \psi}{\partial x^2} = 0 \qquad (2.11)$$

Equation (2.10) is standard but (2.11) is unusual and has some interesting features. If ψ is independent of $\underline{t}$ it reduces to

$$\frac{\partial}{\partial x} \left[\nabla^2 \psi - R(\beta - 1) \frac{\partial \psi}{\partial x} \right] = 0 \qquad (2.12)$$

of which one solution is independent of x and the others satisfy an equation reminiscent of the vorticity equation in Oseén's approximation to the Navier Stokes equations of classical hydrodynamics. The existence of a solution independent of x means that there is a possibility that a wake may develop ultimately extending to infinity, which in turn may mean that it is not uniformly valid to impose uniform conditions at infinity. Let us examine how this solution arises by looking at the variation of ψ with t. It is convenient to take the Laplace transform of (2.11) with respect to t, using $\underline{s}$ as new parameter and denoting the transformed function by $\bar{\psi}$. The equation satisfied by $\bar{\psi}$ is

$$\left(\frac{\partial}{\partial x} + s\right) \nabla^2 \overline{\psi} - R\left(\frac{\partial}{\partial x} + s\right)^2 \overline{\psi} + R\beta \frac{\partial^2 \overline{\psi}}{\partial x^2} = 0 \qquad (2.13)$$

of which solutions can be found by writing

$$\overline{\psi} = \int_0^\infty B(k, s) e^{\theta x} \left\{ \begin{matrix} \sin ky \\ \cos ky \end{matrix} \right\} dk \qquad (2.14)$$

where k is real and $\underline{\theta}$ such that the real part of θx is negative. From (2.13) θ satisfies

$$(\theta + s)(\theta^2 - k^2) - R(\theta + s)^2 + R\beta\theta^2 = 0 \qquad (2.15)$$

and we take $\underline{s}$ to be real and positive. When $\theta = -\infty$ the left hand side of (2.15) is negative, when $\theta = -s$ it is positive and equal to $R\beta s^2$, when $\theta = 0$ it is negative and equal to $-(sk^2 + Rs^2)$ and finally when $\theta = +\infty$ it is positive. Thus there are three real roots θ_1, θ_2, θ_3 of (2.15) which satisfy

$$\theta_1 \leq -s \leq \theta_2 \leq 0 \leq \theta_3 \qquad (2.16)$$

the equality signs being possible only if $s = 0$, $\beta = \infty$, or $R = \infty$, or if

$\beta = 0$ and $s = k$. Otherwise the roots are distinct. Further of the roots only θ_3 is relevant to large positive values of x and θ_1, θ_2 to large negative values of x, since ψ cannot be exponentially large at infinity.

When s is large (2.16) reduces to

$$(\theta + s)(\theta^2 - Rs) = 0 \qquad (2.17)$$

so that $\theta_1 \simeq -s$, $\theta_2 \simeq -\sqrt{Rs}$, $\theta_3 \simeq +\sqrt{Rs}$ as $s \to \infty$. These roots may be interpreted in terms of solutions of the equation for ψ. If $\theta = -s$ when s is large it follows from (2.14) that

$$\psi \simeq F(x - t, \ y) \qquad (2.18)$$

when $x - t$ is small, where F is some function at present arbitrary, and corresponds to a disturbance travelling downstream with the main stream velocity. Such a solution might have been anticipated since the linearization of the equation is partly equivalent to assuming that convection is taking place with the main stream velocity rather than with the local stream velocity. If $\theta = -\sqrt{Rs}$ it follows from (2.14) that

$$\psi \sim g(x, \ y, \ t)\, e^{-Rx^2/4t} \qquad (2.19)$$

when t is small, where g is also arbitrary. This solution corresponds to a diffusion and arises if on setting the fluid into motion the change in the magnetic field due to the currents engendered leads to an initial discontinuity in either of its components at the surface of the body. The discontinuity begins to diffuse in a manner characterized by (2.19).

We now enquire what happens to these solutions as $t \to \infty$ assuming that the ultimate motion is steady. The assumption is equivalent to assuming that $\underline{\theta}$, $\underline{B}$ are regular functions of $\underline{s}$ in the half plane $\underline{re\ s} > 0$ and that on the imaginary axis of s the singularities, if any, are sufficiently weak, except at $s = 0$, for their contribution to ψ to die out algebraically as $t \to \infty$. It is then sufficient to consider the behaviour of θ near $s = 0$. When $s = 0$ (2.1) reduces to

$$\theta(\theta^2 - k^2) + R(\beta - 1)\theta^2 = 0 \ ; \tag{2.20}$$

from the definition of θ_1, θ_2, θ_3 in (2.16) it follows that

$$
\left.
\begin{aligned}
\theta_1 &\to -\frac{1}{2} R(\beta - 1) - \frac{1}{2}\sqrt{R^2(\beta - 1)^2 + 4k^2} \\[2ex]
\theta_2 &\to 0 \\[2ex]
\theta_3 &\to -\frac{1}{2} R(\beta - 1) + \frac{1}{2}\sqrt{R^2(\beta - 1)^2 + 4k^2}
\end{aligned}
\right\}
\tag{2.21}
$$

as $s \to 0$. More precisely as $s \to 0$

$$\theta_2/s \to -1 \qquad\qquad (2.22)$$

One may reasonably suppose that the correct form for ψ in a particular problem is compounded of multiples of the three solutions defined by (2.14)(2.15) and further that that part of the disturbance characterized by ψ is centred at the body at $t = 0$, moving away from it as t increases. It then follows that only the solution ψ_3 given by (2.13) with $\theta = \theta_3$ affects the region upstream of the body (where x is negative), that when t is small ψ_3 is diffusive and that when t is large ψ_3 decays exponentially with x for finite R and $\beta \neq 1$. Hence at large distances upstream the perturbation in the state of the fluid is dominated by the harmonic term ϕ. Downstream of the body the solutions ψ_1, ψ_2 with $\theta = \theta_1$, θ_2 are relevant and a comparison between (2.17)(2.21)(2.22) shows their rôles interchange as t increases from 0 to ∞. When t is small ψ_1 is wave-like while ψ_2 is diffusive. On the other hand when t is large ψ_1 decays exponentially with x while ψ_2 is a function of $x-t$ and y. In fact since we are assuming steady motion we have from (2.4), (2.8) that $\partial\psi_2/\partial x$ is independent of x and ψ_2 has the form

$$(x-t)F(y).$$

Of the solutions derived ψ_1, ψ_3 also satisfy

$$\nabla^2 \chi + R(\beta - 1)\frac{\partial \chi}{\partial x} = 0 \qquad\qquad (2.23)$$

and for convenience we shall denote the x-derivative of that part of ψ which

satisfies (2.23) by Ψ . The other solution ψ_2 is, from continuity, only

zero inside a region C bounded by straight lines parallel to the x axis

and by infinity on the downstream side. The fourth part of the boundary can

be defined when the body is thin: it consists of the rear portion of the body

and the straight lines mentioned above touch the body.

If the body is not thin the linearized equations are in general not valid

in the neighborhood of the body, so that C cannot be fully defined. However

the description of ψ as $t \to \infty$ given above can be used to discuss the ultimate

flow pattern at large distances from the body. It will consist of a harmonic

term and a downstream wake for all values of R, β . In addition if R is

large there is a wake-like contribution from Ψ extending downstream in a

region somewhat like C if $\beta < 1$ and extending upstream in an otherwise

similar region if $\beta > 1$. In fact if $R = \infty$, $\beta < 1$, Ψ is of the same form as

$\partial \psi_2 / \partial x$ when x is large and positive and if $\beta > 1$, Ψ is independent of x

when x is large and negative. It is noted that ψ_2 makes no contribution

to $\hbar$ in virtue of (2.8) so that at finite R, β only the velocity distribution

can be non-uniform in C . On the other hand $\hbar$ can be non-zero in the wakes

produced by Ψ when $R = \infty$.

Let us now examine the properties of ψ in the two limiting cases

$R \to 0$ and $R \to \infty$. First if $R = 0$ (2.15) reduces to

$$(\theta + s)(\theta^2 - k^2) = 0 \qquad\qquad (2.24)$$

with roots $\theta = -s$ and $\theta = \pm k$ and here the root which behaves like θ_1 when s is large behaves like θ_2 when s is small and vice-versa. There is no interchange of roots at $s = k$ when (2.24) has a double root as may be seen by tracing the roots when $\arg s$ is small and positive. Thus the solution in the limit $R \rightarrow 0+$ is not the same as the solution when $R = 0$ and so care must be exercised in inferring flow properties when R is small from those when $R = 0$. Sketches of the behaviour of θ_1, θ_2 regarded as functions of $\underline{s}$ when $R = 0$ and when $0 < R \ll 1$ are given in Figure 1.
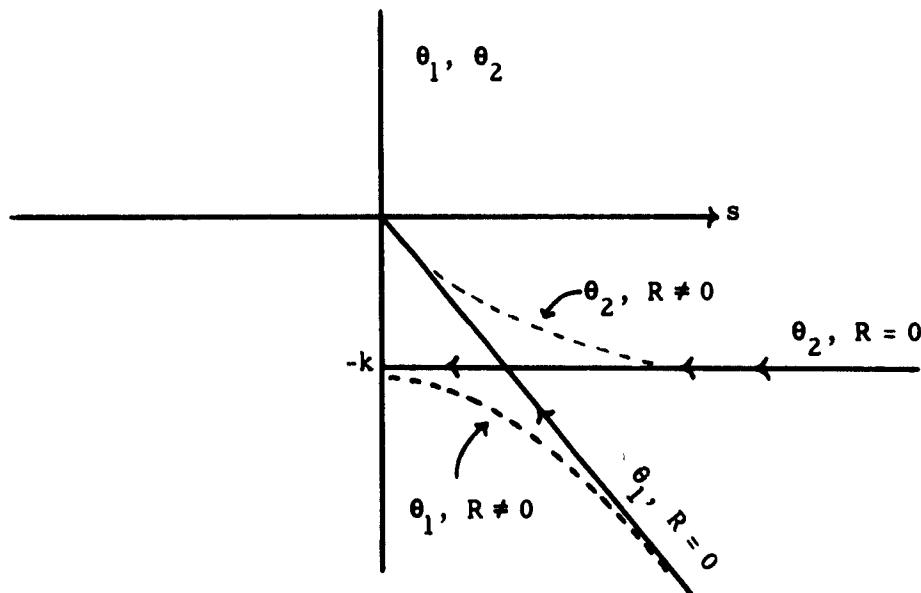
FIGURE 1

Second if $R = \infty$ (2.15) reduces to

$$(\theta + s)^2 = \beta \theta^2$$

i.e.

$$\theta = s(\pm \beta^{\frac{1}{2}} - 1)^{-1} .$$

(2. 25)

These two solutions correspond to θ_1, θ_2 if $\beta < 1$ and to θ_2, θ_3 if $\beta > 1$. The third root of (2.15) becomes $+\infty$ if $\beta < 1$ and $-\infty$ if $\beta > 1$. On the other hand if R is large but finite $\theta_2/s \to -1$ as $s \to 0$ which is different from (2.25). A more careful scrutiny of (2.15) shows that when R is large and s is small the form taken by the roots depends crucially on Rs. Take $\beta < 1$ and then $\theta_3 = R(1-\beta)$ which is effectively infinite. The other two roots satisfy

$$-k^2(\theta + s) = R(\theta + s)^2 - R\beta\theta^2$$

(2. 26)

with solution

$$\frac{\theta}{s} = - \frac{k^2 + 2Rs \pm \sqrt{k^4 + 4\beta k^2 Rs + 4\beta(Rs)^2}}{2(1-\beta)Rs} .$$

(2. 27)

Using the inversion formula for Laplace transforms we can now describe the behaviour of ψ when t is large. If $\beta < 1$, $1 \ll t \ll R$, the solutions of (2.13) corresponding to θ_1, θ_2 are two Alfvén waves

travelling downstream from the body without dispersion and with velocities given by $1 \pm \sqrt{\beta}$. As t increases through R dispersion and change of speed of propagation occur until when t >>R the flow pattern described in the general case is recovered. Again therefore caution must be exercised in inferring steady flow properties when R is large from those when R is infinite. This point has already been discussed by the author in a particular case (7) to which the reader is referred for further details.

3. Application To Flow Past Thin Bodies

The linearized theory of magneto-fluid dynamics may be used to study the complete flow field in two circumstances: when the body is thin and orientated so that the streamlines are almost straight and when the magnetic field is strong ($\beta \gg 1$) so that the inertia terms are negligible. In this section we consider the first of these deferring the second to §4 . Suppose the body occupies the neighborhood of that part of the x axis for which $|x| < 1$. From the theory of §2 it follows, on assuming that the ultimate flow field is steady and almost uniform, that in the fluid

$$v_x = \frac{\partial \Phi}{\partial x} + \beta \frac{\partial \Psi}{\partial y} + \boxed{F(y)} \ , \qquad v_y = \frac{\partial \Phi}{\partial y} - \beta \frac{\partial \Psi}{\partial x} \ \Bigg\}$$

$$h_x = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y} \qquad , \qquad h_y = \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x} \Bigg\} \qquad (31)$$

where we have written $\underline{\Phi}$ for $\partial \phi / \partial x$, Ψ satisfies (2. 23) and both grad Φ, grad Ψ vanish at infinity: the box surrounding F(y) in the equation for v_x means that there is an additional contribution to v_x , independent of x, inside C_+ , a region bounded by the rear part of the body and two lines parallel to the x axis which touch the body and

extend to infinity downstream of it (See Figure 2).

Direction of Flow
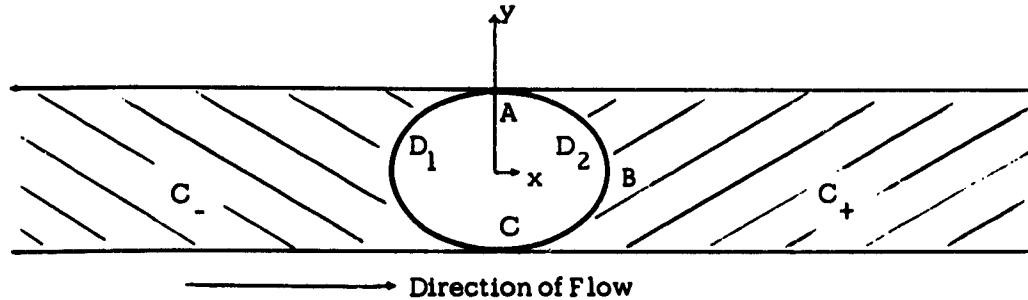
FIGURE 2

Inside the body the velocity field is zero and the velocity field is
harmonic. As a consequence the variation in the magnetic field across the
body is small and since the tangential component of the magnetic field and
the normal component of the induction are continuous at the surface of the
body, it follows that for fixed x the values of h_x, h_y on the top and
bottom surfaces of the body are equal. A further condition to be satisfied
on the body is that the velocity be tangential to the surface: if the surfaces
of the body are given by $y = f_{\pm}(x)$, the $\pm$ signs denoting the top and bottom,
this condition reduces to

$$v_y = f'_{\pm}(x) \qquad\qquad (3.2)$$

on the body, effectively $y = 0\pm$, $|x| < 1$. There is now sufficient
information formally to complete the determination of Φ, Ψ and this

has been done by Lary (5) in a sufficiently representative number of cases. Lary tacitly assumed that $F(y) \not\equiv 0$ but so long as it is small it can be regarded as a quantity to be found after Φ, Ψ are known and in particular makes no contribution to the forces on the body. Hence his description of these forces is essentially complete. For full details of his solution the reader is referred to his paper but it is noted that for large R a strong wake develops upstream or downstream according as $\beta > 1$ or $\beta < 1$ and that the drag $\propto R^{\frac{1}{2}}$. This means that the theory becomes invalid for sufficiently large R and he showed that the condition for validity is that $R \ll \epsilon^{-2}$ where ϵ is a representative thickness of the body. There remains the determination of $F(y)$ which since it leads to a downstream wake for all β is the essential link between the solutions with $\epsilon \ll 1$ and those with $\beta \gg 1$.

As we have seen, the presence of $F(y)$ is bound up with the root $\theta = \theta_2$ of the cubic equation (2.14) and corresponds to magnetic diffusion from the surface of the body at $t = 0$. Thus the solution corresponding to $\theta = \theta_2$ is non-zero at any finite time and a further argument is necessary at $t = \infty$ before it can be included. Basing the argument on uniform conditions at infinity is unsatisfactory since the effect upstream of improving such a condition at large positive values of x dies out algebraically in ϕ and exponentially in Ψ so that it cannot affect the flow near the body.

The further condition by which $F(y)$ is determined must be sought in the flow properties near the body and we suggest that the clue to the condition lies in the vorticity. The necessity for a condition on the vorticity in a linearized theory is most easily seen from a study of the flow when the conductivity of the fluid is zero. The governing equations are then

$$\frac{\partial \underline{v}}{\partial t} + \frac{\partial \underline{v}}{\partial x} = \text{grad } P, \qquad \text{div } \underline{v} = 0 \qquad (4.3)$$

from $(2.5), (2.6)$. In two-dimensional motion (4.3) implies that

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} = 0 \qquad (4.4)$$

where ζ is the vorticity. This equation simply means that vorticity is convected along lines parallel to the x-axis. On assuming that all vorticity originates at the body it follows from (4.4) that ζ is zero except inside C_+, which extends downstream from the body. Inside C_+ the behaviour of ζ cannot be found without making some assumption about ζ on that part of the body bounding C_+ {ABC in Figure 2} . Thus if $\zeta = g(y, t)$ there,

$$\zeta = g(y, t - x + X(y))$$

in C_+ where $x = X(y)$ on ABC: as $t \to \infty$

$$\zeta \to g(y, \infty) \qquad (4.5)$$

for all x. When the conductivity of the fluid is zero the additional assumption,

by which g may be found, emerges from a study of the flow properties of a

fluid with small viscosity as follows. It is supposed that the shape of the

body is such that the boundary layer remains attached to it over almost all

of its length. Then over almost the whole width of C_+ the fluid particles

have zero vorticity by Kelvin's circulation theorem since they had none

upstream. Hence $g = 0$ except possibly when $y/Ac \ll 1$.

   If the fluid is conducting (4.4) becomes

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} = \beta \frac{\partial}{\partial x} \left\{ \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} \right\} \tag{4.6}$$

and again ζ can be given an arbitrary value on ABC. In order to fix ζ and

obtain a well-posed problem we adopt a similar argument to that for $\sigma = 0$.

First we suppose that the body is streamlined so that an insignificant amount

of vorticity is shed from the boundary layer. Second we make use of the

magneto-fluid dynamic extension of Kelvin's theorem in a linearized form.

This implies that the vorticity at D_2 may be calculated from the vorticity

at D_1 whose y-coordinate is the same as that of D_2 (see Figure 2) by

integrating (4.6) from D_1 to D_2 using the values of h_x, h_y on the

boundary of the body, i.e. by the values of h_x, h_y as the line $y = 0$.

Here the motion is steady and so the difference in the values of the

vorticity at D_1 and D_2 is equal to

$$- \beta \frac{\partial}{\partial x} \left(\nabla^2 \Psi \right) \Big|_{D_1}^{D_2} \quad .$$

Hence on comparing the values of the vorticity at D_1, D_2 it follows that

$$F(y) \equiv 0$$

and Lary's solution is completely valid if $R \ll \epsilon^{-2}$. It is worth noting in passing that if the vorticity condition had led to a non-zero value of $F(y)$ it would have been $0(\epsilon^2)$ since $F'(y) = 0(\epsilon)$.

4. Discussion

It was pointed out in §1 that the main difference between the unsteady and the finite conductivity approaches to the flow of a perfectly conducting fluid past a body in the presence of an aligned field occurs if $\beta > 1$ and concerns the existence of a velocity wake on the downstream side of the body. It has been shown in §3 that the steady state solution of the problem when $\sigma < \infty$ contains an arbitrary function which gives rise to this wake and is associated with the vorticity shed from the body. Provided the flow remains almost uniform and the body is streamlined to delay boundary layer separation the vorticity shed may be neglected and this function vanishes. The flow pattern is then as predicted by Lary. Thus the uncertainty in the solution is resolved by an appeal to the realism of the flow predicted.

Unfortunately the flow diverges strongly from its uniform state either as $R \to \infty$ or $\beta \to \infty$. If $R = \infty$ and β finite there is a controversy about the character of the flow but for $\beta = \infty$, $R = \infty$ there exist strong magnetic and velocity wakes upstream and downstream of the body ($\underline{3}$). Although the solution with R finite and $\beta = \infty$ ($\underline{7}$) differs from that one because the magnetic wake is absent the two can be reconciled. Consequently we need only consider the discrepancy between the solutions with $\beta \gg 1$ and finite β , R being finite in each case.

Let us now examine the form taken by (3.1) when β is large and R finite. The equation satisfied by Ψ reduces to

$$\frac{\partial \Psi}{\partial x} = 0 \qquad\qquad (4.1)$$

except near the part ABC of the body where a boundary layer type of
phenomenon can develop in Ψ. Assuming that the velocity is of order
one at most, h_x, h_y, v_x, Φ are negligible in (4.1) and

$$v_x = \begin{cases} F_1(y) & \text{in } C_+ \\ F_2(y) & \text{in } C_- \\ 0 & \text{elsewhere} \end{cases} \qquad\qquad (4.2)$$

where F_1, F_2 are functions of y to be found. The solution for $\beta \gg 1$
(7) is obtained by setting $F_1 = F_2 = -1$ so that the velocity is zero in $C_{\pm}$
and undisturbed everywhere else. This form is consistent with the
approximations made and appears to be more realistic than the form
obtained by extending Lary's condition on $F(y)$ for then, $F_1 = 0$ and
$F_2 = -1$: the boundary condition on the velocity over ABC being satisfied
by means of the boundary layer in Ψ mentioned earlier. The alternative
requires motion of the fluid in ABC which a real fluid would be unlikely
to sustain.

There is however some experimental evidence which should be
borne in mind in connection with these problems. The theoretical flow
properties when $\beta \gg 1$ are very similar to those when a body moves
slowly along the axis of a rotating fluid namely that upstream and down-
stream wakes are predicted. Experiments to test this theory have been

carried out by Taylor (8) who examined the flow in the upstream wake and

by Long (9) who examined the flow ahead and behind the body. Both

experiments confirmed the existence of the upstream wake but Long found

that the fluid streamlines closed up behind the body contrary to the theoretical

prediction. No firm explanation of the discrepancy has been offered but a

tentative explanation is as follows: in Long's experiments the body was

moved under gravity in a finite cylinder of fluid and it is possible that the

incipient motion of the fluid in C_+ , in line with the theory, caused a

cavity at a external boundary of the fluid, the filling up of which greatly

disturbed the fluid properties in C_+ . It would be of interest to examine

the flow if the body is fixed and a rotating fluid forced past it to see whether

the fluid still closes up in C_+ . From the point of view of the present paper

Long's experiment, while making us optimistic that the wake in C_- would

be obtained experimentally, must make us cautious about accepting the

solution with $F_1 = -1$. It seems in fact that further theoretical progress

depends on obtaining experimental evidence of the flow properties at large

values of R.

PART II

APPROXIMATELY ALIGNED MAGNETIC FIELD

5. Formulation

Here we suppose that the motion is steady but that the imposed magnetic field is inclined at a small angle α to the x-axis. Further the body is thin, symmetric and symmetrically disposed to the main stream so that in (3. 2)

$$f_+(x) = -f_-(x) \ .$$

We assume that the perturbations in the velocity and magnetic fields are small and write for their components relative to the x, y, z axes

$$\underline{q} = V_\infty(1 + v_x, \ v_y, \ 0), \qquad \underline{H} = H_\infty(\cos\alpha + h_x, \ \sin\alpha + h_y, \ 0) \tag{5.1}$$

and on substituting into (2. 1)(2. 2) a set of linear equations is obtained after neglecting squares and products of small quantities. It may then be shown that $\underline{v}$, $\underline{h}$ can be expressed in terms of two functions ϕ, ψ viz:

$$\left.\begin{aligned}
v_x &= \frac{\partial^2\phi}{\partial x^2} + \beta\left[\cos\alpha\,\frac{\partial^2\psi}{\partial x\partial y} + \sin\alpha\,\frac{\partial^2\psi}{\partial y^2}\right] \\[2em]
v_y &= \frac{\partial^2\phi}{\partial x\partial y} - \beta\left[\cos\alpha\,\frac{\partial^2\psi}{\partial x^2} + \sin\alpha\,\frac{\partial^2\psi}{\partial x\partial y}\right]
\end{aligned}\right\} \tag{5.2}$$

$$h_x = \cos \alpha \frac{\partial^2 \phi}{\partial x^2} + \sin \alpha \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} \quad ,$$

$$h_y = \cos \alpha \frac{\partial^2 \phi}{\partial x \partial y} - \sin \alpha \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x^2} \quad ,$$

(5.3)

where ϕ is harmonic and

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = R \frac{\partial^2 \psi}{\partial x^2} - \beta R \left(\cos^2 \alpha \frac{\partial^2 \psi}{\partial x^2} + 2 \cos \alpha \sin \alpha \frac{\partial^2 \psi}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2 \psi}{\partial y^2} \right). \quad (5.4)$$

It follows that

$$\frac{p - p_\infty}{\rho V_\infty^2} = - \frac{\partial^2 \phi}{\partial x^2} + \beta (-\cos \alpha \frac{\partial^2 \psi}{\partial x \partial y} + \sin \alpha \frac{\partial^2 \psi}{\partial x^2}) . \quad (5.5)$$

The equation (5.4) satisfied by ψ is superficially somewhat different from (2.11) but they are reconcilable if regarded as particular cases of a more general equation in which $\alpha \neq 0$ and ψ depends on t. We shall see however that the non-uniqueness obtained in Part I is absent in the present investigation. The reason for this difference is not entirely clear and further work on this aspect of the problem would seem desirable.

As in earlier work (10) we make use of Fourier transforms to get a general solution of ψ in a form convenient to our purposes. Write

$$\phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_+(\omega) e^{i\omega x - \xi y} d\omega \qquad (5.6)$$

$$\psi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_+(\omega) e^{i\omega x - \chi_+ y} d\omega \qquad (5.7)$$

in the half plane $y > 0$ where Φ_+, Ψ_+ are arbitrary functions. From the differential equations satisfied by ϕ, ψ

$$\xi^2 = \omega^2 + O^2 \qquad (5.8)$$

$$[i\omega + \beta R \sin^2 \alpha]\chi_+^2 - 2i\beta R\omega \cos \alpha \sin \alpha \, \chi_+ + \omega^2 R - \omega^2 \beta R \cos^2 \alpha - i\omega^3 = 0 \qquad (5.9)$$

and to ensure convergence of the integrals ξ, χ_+ must have positive real parts on the real axis of ω. So far as ξ is concerned this means that the upper half plane $\text{re } \omega > 0$ must be cut along the positive imaginary axis, the lower half plane $\text{re } \omega < 0$ must be cut along the negative imaginary axis and ξ chosen to be real and positive on the real axis of ω. The specification of χ_+ requires a little more care. From (5.9)

$$\chi_+ = \frac{i\omega\beta R \cos \alpha \sin \alpha \pm i\sqrt{\omega^2[\omega^2 - i\omega R(\beta - 1) + \beta R^2 \sin^2 \alpha]}}{i\omega + \beta R \sin^2 \alpha} \qquad ; \qquad (5.10)$$

when ω is large $\chi_+ = \pm \dfrac{\sqrt{\omega^4}}{\omega}$ and it follows that $\sqrt{\omega^4}$ must be $\omega|\omega|$ being positive and negative with ω. Hence the ω plane must be cut at infinity both above and below the real axis and the positive sign chosen in (5.10). For convenience the cuts may be made along the imaginary axis and therefore they extend inwards towards the origin as far as the first zeroes met of the expression under the square root sign in (5.10) i.e. as far as is_1, $-is_2$ where s_1, $-s_2$ are the positive and negative roots of

$$s^2 - R(\beta - 1)s - \beta R^2 \sin^2 \alpha = 0 \ .$$

From these points inwards to the origin the cuts are absent until the next zeroes are reached and since these both occur at the origin the factor $\sqrt{\omega^2}$ may be replaced by ω. Thus

$$\chi_+ = \frac{i\omega\left(\beta R \cos \alpha \sin \alpha + \sqrt{\omega^2 - i\omega R(\beta - 1) + \beta R^2 \sin^2 \alpha}\right)}{i\omega + \beta R \sin^2 \alpha} \ . \qquad (5.11)$$

If $R = \infty$

$$\chi_+ = \frac{i\omega}{\sin \alpha}\left(\cos \alpha + \beta^{-\frac{1}{2}}\right) = i\omega c_+ \qquad (5.12)$$

which means that the corresponding form for ψ is

$$F_+(x - c_+ y), \quad y > 0 \tag{5.13}$$

where F_+ is an arbitrary function, in agreement with earlier work (5).

This solution represents an Alfvén wave extending to infinity in a direction

between the magnetic field and the downstream directions. If $\alpha = 0$, i.e.

the magnetic field is aligned

$$\chi_+ = (\omega^2 - i\omega R(\beta - 1))^{\frac{1}{2}} \tag{5.14}$$

the cut extending over the whole imaginary axis except for the strip

$0 < i|\omega| < iR|\beta - 1|$. Comparison of (5.13) with (5.15) indicates that

the case $\alpha = 0$ is not a straight forward limit of the solution when $\alpha \neq 0$.

The singular nature of the limit is also brought out by studyir.g the double

limit procedure $\alpha \to 0$, $R \to \infty$ holding $R\alpha^2 = \theta^2$ fixed. The expression

for χ_+ is then

$$\frac{\chi_+}{R^{\frac{1}{2}}} = \frac{i\omega}{i\omega + \beta\theta^2} \left\{ \beta\theta + \sqrt{\beta\theta^2 - i\omega(\beta - 1)} \right\} \tag{5.15}$$

showing how sensitive the form for χ_+ is to the value of θ.

Parallel arguments to the above may be used to show that when

$y < 0$

$$\left. \begin{array}{l} \phi(x,\, y) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \Phi_{-}(\omega)\, e^{i\omega x + \xi\, y}\, d\omega \\[2em] \psi(x,\, y) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \Psi_{-}(\omega)\, e^{i\omega x + \chi_{-} y}\, d\omega \end{array} \right\} \qquad (5.16)$$

where Φ, Ψ are arbitrary functions and

$$\chi_{-} = \frac{i\omega}{i\omega + \beta R \sin^2 \alpha} \left[-\beta R \cos \alpha \sin \alpha + \sqrt{\omega^2 - i\omega R(\beta - 1) + \beta R^2 \sin^2 \alpha} \right] . \qquad (5.17)$$

If $R = \infty$

$$\chi_{-} = \frac{i\omega}{\sin \alpha}\, (-\cos \alpha + \beta^{-\frac{1}{2}}) = i\omega c_{-} \qquad (5.18)$$

leading to a corresponding form for ψ,

$$F_{-}(x + c_{-} y) \qquad (5.19)$$

which is different from (5.13). If $\alpha = 0$ however

$$\chi_{-} = [\,\omega^2 - i\omega R(\beta - 1)\,]^{\frac{1}{2}} \qquad (5.20)$$

which is identical with (5.14). Thus if $\beta > 1$, $\alpha \neq 0$, $R = \infty$ the solutions for ψ is $y > 0$, $y < 0$ correspond to wakes roughly pointing <u>downstream</u> and <u>upstream</u>. But if $\beta > 1$, $\alpha = 0$, $R < \infty$ the two solutions are of similar

type. Finally if $\alpha \to 0$, $R \to \infty$, $R\alpha^2 = \theta^2$

$$\frac{X_-}{R^{\frac{1}{2}}} = \frac{i\omega}{i\omega + \beta\theta^2} \left(-\beta\theta + \sqrt{\beta\theta^2 - i\omega(\beta - 1)} \right)$$

(5.21)

which is different from (5.15) for all $\theta > 0$, but not if $\theta = 0$.

In consequence we can expect the wake structure of the flow when $\alpha = 0$ to be of a different kind from the wake structure when $\alpha \neq 0$, even if $\alpha \sim R^{-\frac{1}{2}}$ so that in some sense the limit $\alpha \to 0$ is singular. In the next section the properties of the solution when $\alpha \ll 1$, $R \gg 1$ but $\alpha R^2 \sim 1$ will be explored; it will be found that the singular nature of the limit is confined to the wake structure, the form on the body when $\alpha = 0$ being deducible from the form when $\alpha \neq 0$.

6. Approximately Aligned Field: Symmetric Body

In view of the difference between the wakes for $\beta > 1$ predicted in [4] when the magnetic field is oblique and $\sigma = \infty$, and Lary's theory, it is of interest to examine the properties of the solution when R is large and α, the angle between the undisturbed velocity and magnetic fields, is small. It will be supposed, following the discussion in §5 that the orders of magnitude are such that $R\alpha^2 = \theta^2$ is of order unity. Let

$$[\frac{\partial\phi}{\partial x}] = P(x), \quad [\frac{\partial\phi}{\partial y}] = Q(x), \quad [\frac{\partial\psi}{\partial x}] = S(x), \quad [\frac{\partial\psi}{\partial y}] = T(x) \qquad (6.1)$$

where the symbol [ ] denotes the leap in the function as y increases from 0- to 0+ . If we denote the transforms of P, Q, S, T by bars,

$$\Phi_+ = \frac{\overline{P}}{2i\omega} - \frac{\overline{Q}}{2\xi}, \qquad \Phi_- = -\frac{\overline{P}}{2i\omega} - \frac{\overline{Q}}{2\xi} \qquad (6.2)$$

$$\Psi_+ = \frac{\overline{S}\chi_-}{i\omega(\chi_+ + \chi_-)} - \frac{\overline{T}}{\chi_+ + \chi_-}, \qquad \Psi_- = -\frac{\overline{S}\chi_+}{i\omega(\chi_- + \chi_+)} - \frac{\overline{T}}{\chi_+ + \chi_-} \qquad (6.3)$$

these formulae holding for general values of α, R. We now specialize to small values of α and then, from (5.3), since h_x, h_y are continuous at $y = 0$

$$\left.\begin{array}{c} \overline{P} + \alpha\overline{Q} + \overline{T} = 0 \\ \\ \alpha\overline{P} - \overline{Q} + \overline{S} = 0 \end{array}\right\} \qquad (6.4)$$

Further if we write

$$\overline{v}_y = F(\omega)\ \text{sgn}\ y + F_1(\omega) \qquad (6.5)$$

at $y = 0$ where $F(\omega)$ is the Fourier transform of a function equal to $f'_+(x)$ if $|x| < 1$ and is otherwise zero, while $F_1(\omega)$ is the Fourier transform of a function which is arbitrary in $|x| < 1$ and zero in $|x| < 1$, (5.2) yields

$$F(\omega) = \frac{1}{2}\ i\omega\overline{S}(1 - \beta)\ , \qquad (6.6)$$

$$F_1(\omega) = \frac{1}{2}\ \xi\overline{T} + \overline{S}\left[\frac{\alpha\beta x_- x_+}{x_+ + x_-} - \frac{i\omega\beta(x_- - x_+)}{2(x_+ + x_-)}\right] \qquad (6.7)$$

using (6.4) and retaining only leading terms. From (6.6) we have

$$S'(x) = \frac{2}{1 - \beta}\ f'_+(x) \qquad |x| < 1 \qquad (6.8)$$

and otherwise is zero, in agreement with the corresponding result in (5). Finally since the pressure is continuous on that part of the x-axis which does not include the body we have

$$i\omega(\overline{P} - \alpha\beta\overline{S} + \beta\overline{T}) = 2G(\omega) \tag{6.9}$$

where $G(\omega)$ is the transform of a function which is zero outside $|x| = 1$.
There is now available sufficient information to determine all unknown
functions in terms of $f_+(x)$ apart from an indeterminancy associated with
circulation and usually removed by a suitable Kutta condition. If in particular
$R \gg 1$, $\alpha \ll 1$ and θ is of order unity χ_+, χ_- are given by (5.15),
(5.21) and since from (6.4)(6.9) $T'(x)$ is non-zero only if $|x| < 1$,
(6.7) may be interpreted as an integral equation

$$\int_{-1}^{+1} \frac{T'(x')\,dx'}{x-x'} = \frac{-2\beta\theta(1+\beta)}{[(\beta-1)^3\pi]^{\frac{1}{2}}} \int_x^1 \frac{S'(x')\,dx'}{(x'-x)^{\frac{1}{2}}} \exp - \left\{ \frac{\beta\theta^2}{\beta-1}(x'-x) \right\} \tag{6.10}$$

giving $T'(x)$ in terms of $S'(x)$ apart from a complementary function of the
form $C(1-x^2)^{-\frac{1}{2}}$ where C is an arbitrary constant presumable determined
by a suitable Kutta condition. It is noted that from (6.10) T' and S' are
of the same order of magnitude and that $T' \to 0$ as $\theta \to 0$ in agreement with
Lary (5).

We are now in a position to compute the force on the body and to
comment on the wake structure. Restricting attention to $\beta > 1$ the drag
on the body is

$$\int_{-1}^{+1} \alpha f'_+(x)\,dx\,(\Delta p_+ + \Delta p_-)\,dx \tag{6.11}$$

where $\Delta p_{\pm}$ are the excess pressures on the top $(y = 0+)$ and bottom surfaces of the body. Using the formula (5.5) for the pressure in terms of ϕ, ψ and (6.2), (6.3) it is fairly easy to show that the Fourier transform of $\Delta p_+ + \Delta p_-$ is

$$\frac{2\beta\rho V_\infty^2 R^{\frac{1}{2}} F(\omega)}{(\beta\theta^2 - i\omega(\beta-1))^{\frac{1}{2}}} \tag{6.12}$$

whence

$$\Delta p_+ + \Delta p_- = \frac{2\beta\rho V_\infty^2 \theta}{\alpha\{\pi(\beta-1)\}^{\frac{1}{2}}} \int_x^1 \frac{f'_+(x')\,dx'}{(x'-x)^{\frac{1}{2}}}\, \exp - \left\{\frac{\beta\theta^2}{\beta-1}(x'-x)\right\} \tag{6.13}$$

and (6.11) may now be evaluated in any specific case. At large values of θ (6.13) reduces to $2f'_+(x)/\alpha$ in agreement with (4) and as $\theta \to 0$, (6.13) reduces to

$$\frac{\rho V_\infty^2 \beta}{\pi}\left(\frac{\pi R}{\beta-1}\right)^{\frac{1}{2}} \int_x^1 \frac{f'_+(x')\,dx'}{(x'-x)^{\frac{1}{2}}} \tag{6.14}$$

in agreement with Lary (5).   From the formula for the excess pressure it

is possible to deduce a condition for the validity of the assumption that

the linearized equation and boundary conditions may be used, namely

$$\text{Max} \left\{ R^{\frac{1}{2}}, \frac{1}{\alpha} \right\} << \frac{1}{\epsilon} \tag{6.15}$$

where ϵ is a measure of the thickness of the body.

    Let us now look at the structure of the wake when $\beta > 1$, focusing

attention on the downstream wake, present if $\theta = \infty$ but absent if $\theta = 0$.

The upstream wake and the case $\beta < 1$ can be examined by similar methods.

The downstream wake is given by $\partial^2 \psi / \partial x \partial y$ for it makes the dominant

contribution to p, v_x in the range of values of R, α of interest here.   The

wake only appears if $y > 0$ when

$$\frac{\partial' \psi}{\partial x \partial y} = - \frac{R^{\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} \frac{F(\omega) \, d\omega}{\{\beta\theta^2 - i\omega(\beta-1)\}^{\frac{1}{2}}} \, \exp\left[i\omega x - \frac{\omega R^{\frac{1}{2}} y}{\omega - i\beta\theta^2} \{\beta\theta + \sqrt{\beta\theta^2 - i\omega(\beta-1)} \} \right]. \tag{6.16}$$

As $\theta \to \infty$ this expression reduces to

$$- \frac{1}{2\pi\alpha\beta^{\frac{1}{2}}} \int_{-\infty}^{\infty} F(\omega) \, d\omega \, \exp i\omega \{x - c_+ y\}$$

$$\tag{6.17}$$

$$= - \frac{1}{\alpha\beta^{\frac{1}{2}}} \, f'_+(x - c_+ y)$$

if $|x - c_+ y| < 1$ and is otherwise zero, where c_+ is defined in (5.12).
This result is in agreement with (4) and shows that when θ is large the
wake is controlled by the exponential term. At a finite value of θ, $x > 1$,
and taking $f_+(x)$ to be a polynomial, the contour of integration in (6.16)
may be deformed into a finite closed curve C surrounding $\omega = i\beta\theta^2$.
As $\theta \to 0$ this contour can become a point circle at the origin so that
the integral vanishes. When $\theta << 1$ only the behaviour of the integrand
near $\omega = 0$ is relevant and since $f_+(1) = f_+(-1)$ the limit of $F(\omega)/i\omega$ as
$\omega \to 0$ exists and we write it as A. Consequently (6.16) is equivalent to
writing

$$\frac{\partial^2 \psi}{\partial x \partial y} = A\theta^3 R^{\frac{1}{2}} Y \mathcal{H}(X, Y) \tag{6.18}$$

if $\theta << 1$ where $X = \theta^2 x$, $Y = \theta R^{\frac{1}{2}} y$ and

$$\mathcal{H}(X, Y) = -\frac{1}{2\pi Y} \int_C \frac{S \, ds}{\{\beta - is(\beta - 1)\}^{\frac{1}{2}}} \exp\left\{isX - \frac{sY}{s - i\beta}(\beta + \sqrt{\beta - is(\beta - 1)}\right\}.$$

If Y is small, $\mathcal{H}$ may be expanded as a Maclaurin series in Y whose
coefficients are functions of X; the leading terms are

$$2\beta^2 e^{-\beta X} + 2\beta^3 Y e^{-\beta X}(\beta X - 3) + O(Y^2). \tag{6.19}$$

If Y is large $\mathcal{H}$ may be evaluated by the method of steepest descent and we have

$$\simeq \frac{(X-\tau Y)\beta^{\frac{1}{4}}}{\tau^3 (2\pi Y^5)^{\frac{1}{2}}} \exp -\left\{ \frac{(X-\tau Y)^2}{2\tau^2 Y} \right\} \qquad (6.20)$$

where $\tau = 1 + \beta^{-\frac{1}{4}}$. Thus the strength of the wake falls off rapidly as $\theta \to 0$ from (6.18) and in addition becomes much more diffuse X, Y remaining small for a wide range of values of x, y . Nevertheless the characteristic wake structure, albeit in a very weakened form, remains for all $\theta > 0$. This follows from (6.20) which shows that the wake is centred on the line $X = \tau Y$ i.e. $x = c_+ y$ and spreads out like $Y^{\frac{1}{2}}$ far downstream from the body. It is noted that (6.20) represents the wake structure when Y is large for all non-zero θ including $\theta = \infty$ and consequently shows how the structure depends on θ . For example when θ is large the thickness of the wake $\sim y^{\frac{1}{2}}/\theta^{\frac{1}{2}}$ as would be expected since the wake has a constant finite thickness when $\theta = \infty$.

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